Steady state behavior of mechanically perturbed spin glasses and ferromagnets

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A zero temperature dynamics of Ising spin glasses and ferromagnets on random graphs of finite connectivity is considered. Like granular media, these systems have an extensive entropy of metastable states. We consider the problem of what energy a randomly prepared spin system falls to before becoming stuck in a metastable state. We then introduce a tapping mechanism, analogous to that found in real experiments on granular media. This tapping, corresponding to flipping each spin with probability p simultaneously, leads to a stationary regime with a steady state energy $E(p)$. We explicitly solve this problem for the one-dimensional ferromagnet and the $\pm J$ spin glass, and carry out extensive numerical simulations for spin systems of higher connectivity. In addition our simulations on the ferromagnetic systems reveal a first order transition, whereas the usual thermodynamic transition on these graphs is second order.

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I. INTRODUCTION

Recently there has been much experimental and theoretical interest in the properties of granular media. In such systems the thermal energy available is not sufficient to allow the rearrangement of a single particle, and hence the system is effectively at zero temperature in the thermal sense. The fact that the problem is not trivial lies in the fact that such systems have an exponentially large number of such metastable states, which may be also called blocked or jammed configurations. Edwards associated an entropy with these configurations,

$$S_{Edw}=\ln(N_{MS}), \tag{1}$$

where N_{MS} is the total number of metastable states of the system [1]. It is reasonable to assume that in complex systems such as granular media S_{Edw} is extensive, meaning that $N_{MS}=\exp(Ns)$ where s is the entropy per particle which in the thermodynamics limit becomes independent of N. Alternatively one may work with an entropy per unit of volume which is clearly a more natural choice in granular media. Because the system has an extensive number of blocked configurations, if it is prepared from a random initial state it will lower its energy via only energy lowering rearrangements until it becomes stuck in a metastable state. Normally the first encountered blocked state will not be that of lowest energy (or most dense packing). In order to change the state of the system an external perturbation such as tapping or shearing is required. In between perturbations the system relaxes into new configurations. A natural and practically very important question concerning this sort of dynamics is the following: What are the properties of the steady state regime obtained via such mechanical perturbation schemes?

Recently it was shown that spin glasses and ferromagnets on random graphs have an extensive entropy of metastable states, and the one may calculate this entropy at fixed values of the energy [2,3]. Therefore, though they are quite different physically to granular materials, these systems have an extensive entropy of metastable states as do granular media. The possibility of using spin glasses as a paradigm for granular material was first introduced in Ref. [4].

Let us recall an example of an experiment on a system of hard spheres reported in Ref. [5]. A system of dry hard soda glass spheres is placed in a glass tube. The system is tapped by using a piston to move the tube vertically through a sine cycle. The tapping parameter Γ is defined to be the ratio of the maximal acceleration due to the piston in the cycle to g the acceleration due to gravity. After an initial irreversible curve, obtained by increasing the tapping rate slowly, the system arrives at a reversible curve where the density is a monotonic function of Γ, the highest packing densities being obtained at the lowest tapping rate. Numerical simulations on granular media [6] reveal a similar behavior (though the irreversible part of the experimental curve corresponding to a loosely packed *fluffy* metastable state was not seen). It was also observed that at small tapping the relaxation to the final density is extremely slow, and is well fitted by an inverse logarithmic decay of the form

$$\rho(t)=\rho_\infty-\frac{\Delta\rho_\infty}{1+B\ln(1+t/\tau)}, \tag{2}$$

where ρ_∞ (the final density), $\Delta\rho_\infty$, τ (the characteristic relaxation time), and B are fitting parameters. However, it should be remarked that the behavior of granular systems is strongly dependent on the tapping mechanism, and that horizontal shearing [7] leads to a behavior qualitatively different to vertical tapping.

In this paper we extend and elaborate a preliminary report of the results of Ref. [8]. The philosophy of this paper is to examine spin glasses as paradigms for granular media. Here the quantity corresponding to the density is the energy of the system. We allow the system to evolve under a random sequential zero temperature single spin flip dynamics where only moves which reduce the energy are allowed. When the system is blocked we tap it with strength $p\in[0,1/2]$, that is to say each spin is flipped with a probability p, the updating at this point being parallel. The system is then evolved by the zero temperature dynamics until it once again becomes stuck; the tapping is then repeated. Physically this corresponds to assuming that, in granular media, the relaxation time to a new metastable state is much shorter than the time between taps. A similar, though not identical, tapping dynamics has also been introduced independently in the context of three spin ferromagnetic interactions on thin hypergraphs [9], also with the goal of studying the dynamics of granular media. We find that a stationary regime is reached after a sufficiently large number of taps, characterized by a steady state energy $E(p)$ (analogous to the stationary density, the same analogy as used in Ref. [9]). The initial dynamics from the random initial configuration into the first metastable state is examined analytically for the one-dimensional $\pm J$ spin glass or ferromagnet (the two are equivalent by a gauge transformation). We call this the initial fall, and the average energy of the first metastable state visited E_f is computed. We then develop a mean field theory for the dynamics under falling, then tapping; interestingly, this theory appears to be exact in the case of a one-dimensional system, and one may calculate $E(p)$ within this scheme, the results being in excellent agreement with the numerical simulations.

Numerically we examine the tapping of spin glasses and ferromagnets of higher connectivity. For the spin glass we find that $E(p)$ is, as in the experiments, a decreasing function of p. For small p we define the exponent θ by $E(p) \sim E(0^+) + Ap^\theta$, with A a constant. In the one-dimensional case we show analytically that $E(p) \sim -1 + \sqrt{2p}$, hence $\theta = 1/2$, whereas for spin glasses on thin graphs for connectivity superior to two we find that $\theta = 1$. However, for $p < 0.05$ we find that the time to reach the steady state is extremely long and not accessible numerically. In this slow dynamical regime we find a slow relaxation of the time dependent energy, reminiscent of that observed in experiments on granular media [5], and hence compatible with Eq. (2).

In the case of the ferromagnet we find numerically that there exists a critical value p_c of p such that for $p > p_c$, $E(p) > E_{GS}$ where E_{GS} is the energy of the ground state and the inequality is strict, and that for $p < p_c$ $E(p) = E_{GS}$. Hence in the ferromagnetic system there is a first order phase transition under the tapping dynamics (in contrast to the usual thermodynamic ferromagnetic transition in these systems, which is second order [10]).

There have of course been many models studied to understand the compaction process in granular media [11–13], which reproduced many of the experimental features. Here the spin glass is clearly far from a realistic realization of a granular media; however, the fact that it has an extensive entropy of blocked states and the obviously natural form of the tapping dynamics implemented makes it a natural testing ground for ideas about dynamics and the possible thermodynamics of systems such as granular media. Moreover, it was argued in Ref. [11] that the slow compaction regime is well explained if we assume that particles can rearrange themselves in such a way as to create a particle size void, which is quickly filled by a new grain. This mechanism involves a crossing of energy barriers, and leads to a logarithmic compaction before the asymptotic steady state regime [12]. We expect that the local rearrangements which occur during the tapping dynamics on spin glasses random graphs will lead to a behavior analogous to the slow glassy dynamics of systems as granular media.

Of course one would ultimately like to obtain a theoretical understanding of the asymptotic, steady state regime of lightly tapped granular media. Edwards proposed [1] that a light tapping dynamics on granular type systems leads to a steady state whose properties are determined by a flat measure over the blocked or metastable states satisfying the macroscopic constraints involved e.g., fixed internal energy and compactivity). This idea recently attracted much interest, and was examined in the context of various models [14–17]. In this paper we shall concentrate simply on the asymptotic energy of the final tapped state. A study of the dynamics leading to this final regime is deferred for further investigation [18]. We shall see that the calculation of the Edwards entropy as a function of energy gives us a possible explanation of the first order ferromagnetic transition.

II. SPIN SYSTEMS ON THIN GRAPHS

The models we shall consider are spin systems on random thin graphs. A random thin graph is a collection of N points, each point being linked to exactly c of its neighbors, c therefore being the connectivity of the graph. The distribution of metastable states in these systems was recently considered in Refs. [2,3]. The spin glass and ferromagnet model we shall consider has the Hamiltonian

$$H = -\frac{1}{2}\sum_{j\neq i} J_{ij} n_{ij} S_i S_j, \tag{3}$$

where S_i are Ising spins; n_{ij} is equal to 1 if the sites i and j are connected, and equal to zero otherwise. The fact that the local connectivity is fixed as c imposes the local constraints $\Sigma_j n_{ij} = c$, for all sites i. In the spin glass case J_{ij} are taken from a binary distribution where $J_{ij} = -1$ with a probability 1/2, and $J_{ij} = 1$ with a probability 1/2. In the ferromagnetic case, $J_{ij} = 1$. Here we define a metastable state as a spin configuration where any single spin flip does not increase the energy of the system. Mathematically the total number of these metastable states is expressed as

$$N_{MS} = \mathrm{Tr} \prod_{i=1}^{N} \theta\left(\sum_{j\neq i} J_{ij} n_{ij} S_i S_j\right), \tag{4}$$

where θ is the Heaviside step function. This equation may be understood as follows. The energy change ΔE_i due to the spin flip $S_i \to -S_i$ is given by $\Delta E_i = \Sigma_{j\neq i} J_{ij} n_{ij} S_i S_j$. Hence a configuration is single spin flip stable if all ΔE_i's for that configuration are non-negative.

It should be pointed out here that the definition of metastable states is of course dependent on the dynamics of the system, in contrast with microstates in classical statistical mechanics. Whether or not, in certain cases, the information about the dynamics encoded in the calculation of the entropy of metastable states is enough to allow one to predict the properties of the steady state regime is an open question.

The fact that, in our definition of metastable states, we include the marginal case (where the energy change is zero) implies that here $\theta(x)$, the Heaviside step function, is taken such that $\theta(0)=1$. In the context of granular media, where friction plays an important role, this is a natural choice as one certainly needs a nonzero force in order to make a grain move. With this definition, the total number of metastable states of internal energy E per spin is formally given by

$$N_{MS}(E)=\mathrm{Tr}\prod_{i=1}^{N}\theta\left(\sum_{j\neq i}J_{ij}n_{ij}S_iS_j\right)\delta(H-NE). \tag{5}$$

The corresponding Edwards entropy per spin, at a fixed energy E per spin, is then given by

$$s_{Edw}(E)=\ln(N_{MS}(E))/N. \tag{6}$$

III. ONE-DIMENSIONAL FERROMAGNET AND SPIN GLASS

We remark that by a gauge transformation the one-dimensional ferromagnet and $\pm J$ spin glass are equivalent, and for transparency place ourselves in the context of the ferromagnet. Let us remark that the zero temperature Glauber dynamics of the one-dimensional ferromagnet can be explicitly solved [19]; here a diffusion of the domain walls occurs, and the dynamics is not blocked. In the Glauber case one may close the dynamical equations; however, here such a closure scheme does not seem possible. The zero temperature Kawasaki dynamics (conserving the total magnetization) of the one-dimensional Ising model, where the system can freeze, was solved in Ref. [20].

To solve the dynamics of the one-dimensional ferromagnet we consider the dynamics from the point of view of the bonds. We define a fault of length n to be a sequence of n neighboring adjacent domain walls. The zero temperature dynamics takes place within these faults via the flipping of one of the $n-1$ spins contained between the n domain walls. We define $I_i(n)$ to be the function (taking the value 1 if its argument is true, and 0 otherwise) indicating that, starting from bond i, there are exactly n consecutive domain walls (there being no domain wall on bond $i-1$ and no domain wall on bond $n+i$ but all the intervening bonds have a domain wall). In the initial configuration we take the probability that a given spin is different from its left neighbor (that is to say the probability of a domain being between two spins) to be a. Hence if $a=0$, we have an initially ferromagnetic configuration. If $a=1$, it we have an antiferromagnetic configuration. The case $a=1/2$ corresponds to a completely random configuration of maximal entropy. The total energy of the initial configuration is then given by

$$\mathcal{E}_0=-N+2\sum_i\sum_n I_i(n), \tag{7}$$

as the energy is given by the ground state energy $-N$ plus two times the number of domain walls (excitations). Defining $K_0(n)$ to be the probability that one has n consecutive domain walls starting from a given site [hence $K_0(n)=\langle I(n)\rangle$ where $\langle\cdot\rangle$ indicates the average over the initial conditions], with the initial conditions introduced above one finds

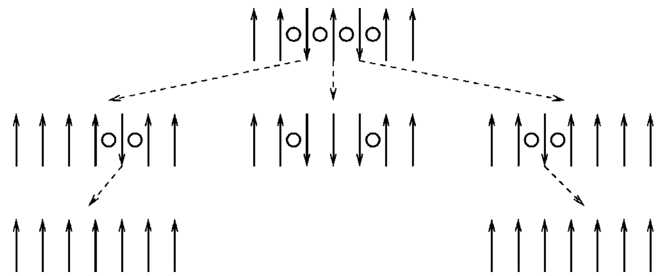

FIG. 1. An example of the falling dynamics for a fault of length 4. The symbols O indicate the presence of a domain wall. The broken arrowed lines start at a spin that is flipped, and lead to the following configuration.

$$K_0(n,N)=(1-a)^2a^n. \tag{8}$$

(The length of the defaults has a geometric distribution.) The initial energy per site of a configuration generated in this manner is therefore (using the translational invariance of the system when $N\rightarrow\infty$) given by

$$E_0=-1+2\sum_{n=1}^{N}K_0(n,N)n. \tag{9}$$

For the distribution of initial configurations considered here, we therefore find that

$$E_0=-1+2a. \tag{10}$$

We define by $\chi(n)$ the average number of isolated domain walls left by a fault of n consecutive domain walls after the zero temperature dynamics described above has finished. The final energy of the system per site E_f is therefore

$$E_f=-1+2\sum_{n=1}^{N}K_0(n,N)\chi(n). \tag{11}$$

By the definition of the spin dynamics, domain walls disappear by pairs of two neighboring domain walls. It is clear that $\chi(1)=1$ and $\chi(2)=0$, and we set $\chi(0)=0$. Within such a fault the dynamics proceeds by flipping one of the $n-1$ spins between the domain walls. This leads to the elimination of the two domain walls on either side of the spin being flipped. An example of this dynamics for a fault of length 4 is shown in Fig. 1. By recurrence, after a random flip we obtain

$$\chi(n)=\frac{1}{n-1}\sum_{k=1}^{n-1}\chi(k-1)+\chi(n-k-1). \tag{12}$$

We solve Eq. (12) by introducing the generating functional

$$g(z)=\sum_{n=1}^{\infty}\chi(n)z^n. \tag{13}$$

The resulting equation for $g(z)$ is

$$z\frac{dg}{dz}=\left(\frac{2z^2}{(1-z)}+1\right)g(z). \tag{14}$$

Solving this with the appropriate boundary conditions, one obtains

$$g(z)=\frac{z\exp(-2z)}{(1-z)^2}, \tag{15}$$

and substituting this into Eq. (11) yields

$$E_f=-1+2(1-a)^2\sum_{n=1}^{\infty}\chi(n)a^n=-1+2(1-a)^2g(a), \tag{16}$$

thus giving the result

$$E_f=-1+2a\exp(-2a). \tag{17}$$

This yields a value of E_f for the completely random initial configuration, where $a=1/2$, of -0.632121. In fact the value of E_f is maximal for the case $a=1/2$. For the totally antiferromagnetic initial condition, where $a=1$, here we find $E_f=-0.72933$. Clearly when $a=0$ the system is already in its ground state, and we find $E_f=-1$ as we should. We note that these values (and those for all a) have been checked with and are in perfect agreement with our numerical simulations. This calculation with the one dimensional ferromagnet demonstrates two important points.

(i) The final value of the energy E_f depends strongly on the initial configuration.

(ii) The system does not fall into a state of energy corresponding to the maximum of $N_{MS}(E)$. In Refs. [2,3] it was shown that $N_{MS}(E)\sim\exp(Ns(E))$ where $s(E)$ is a concave function peaked at $E^*=-1/\sqrt{5}\approx 0.44721$. Hence, even if the total number of metastable states is dominated (in the statistical sense) by those of energy E^*, generic initial conditions always seem to lead to an energy lower than this [9,14]. In Ref. [21] the value of E_f for a variety of zero temperature dynamics (sequential, greedy and reluctant) in the fully connected Sherrington Kirkpatrick (SK) spin glass model [22] was studied, similar behavior was found.

When the one-dimensional system is tapped we find results in line with those described later for the spin glasses at higher connectivity. The curve of $E(p)$, the asymptotic stationary value of the energy at a given p, is shown in Fig. 3 from 100 systems of size 10 000 spins.

The time taken to reach a stationary value for $E(p)$ was rapid for larger p, but for small p there is a very slow relaxation to the final asymptotic state which is of the form $1/\sqrt{t}$, where t is the number of taps. This is easily understood as, at a very slight tapping order, p effects dominate at early times; this shows the following: (i) Isolated pairs of domain walls within large domains are immediately destroyed once tapping is stopped. (ii) Flipping a spin on either side of a domain wall creates domain, wall diffusion and with this annihilation by coalescence of domain walls. Hence the dynamics at small p and early times is qualitatively the same as that for low temperature Ising model coarsening [23].

In order to go beyond our first calculation of E_f and solve the tapping dynamics, we consider a mean field theory for the dynamics of a system of connectivity c. We shall see that at $c=2$ this theory gives the analytical result [Eq. (17)], and reproduces the numerical tapping results to within numerical errors. Once again we concentrate on the dynamics on the bonds. We say that the bond between two connected sites (i,j) is satisfied if it gives a negative contribution to the energy, i.e., $-J_{ij}S_iS_j<0$ (here by definition $n_{ij}=1$ as the sites are connected). In the case of the ferromagnet and $\pm J$ spin glass this contribution to the energy is clearly either 1 or -1. Hence here a bond is satisfied or unsatisfied if its contribution to the energy is -1 or 1. For a given site we define x to be the difference between the number of unsatisfied and satisfied bonds. Hence x is the local field on the spin at this site and $x\in -c,-c+2,\ldots,c-2,c$. If $x>0$ then the spin can flip bringing about the change $x\to -x$. In addition, by $P(x,k)$ we denote the probability that the site of interest has a local field x after a total of k attempted random sequential spin flips under the zero temperature falling dynamics (that is to say the dynamics in between taps). We define f_+ and f_- as the probabilities that a given spin can flip conditional on the fact that the bond with a given neighboring site is not satisfied or satisfied, respectively. Formally we have

$$f_{\pm}=\text{Prob }(x>0|\text{given bond is not satisfied or satisfied}) \tag{18}$$

We may turn around this conditional probability using Bayes' theorem to obtain

$$f_{\pm}=\frac{\text{Prob }(x>0\text{ and given bond is not satisfied or satisfied})}{\text{Prob }(\text{given bond is not satisfied or satisfied})}. \tag{19}$$

Given that a site has local field x, it must have $(c+x)/2$ unsatisfied bonds and $(c-x)/2$ satisfied bonds. Therefore, we find that

$$\text{Prob } (x>0 \text{ and given bond is not satisfied or satisfied}) = \sum_{x>0} P(x)(c\pm x)/2c$$

and

$$\text{Prob } (\text{given bond is not satisfied or satisfied}) = \sum_{x} P(x)(c\pm x)/2c.$$

Putting these results into Eq. (18) then gives

$$f_{\pm}=\frac{\sum_{x>0} P(x)(c\pm x)}{\sum_{x} P(x)(c\pm x)}. \tag{20}$$

If we are interested in the spin at site i the possibilities between time k and $k+1$ are as follows:

(i) The spin at site i is chosen and $x>0$; then the spin at site i will flip and x goes to $-x$.

(ii) The spin at site i is chosen and $x\leq 0$; then the spin at site i cannot flip and x does not change.

(iii) A neighbor of site i with positive local field is chosen and so flips. In this case, x goes to $x+2$ or to $x-2$ depending on whether or not the bond with site i was satisfied or not satisfied.

(iv) A neighbor of site i with negative or zero local field is chosen and so does not flip. In this case, x does not change.

(v) One chooses neither the spin at site i nor any of its neighbors, and so x stays x.

A schematic example of the falling dynamics for a system of connectivity $c=3$ is shown in Fig. 2. Assuming that the distribution at every site is given by $P(x,k)$ and assuming independence between the values of x from site to site (the mean field approximation) we obtain

$$\begin{aligned}P(x,k+1)=&\frac{\theta(-x)}{N}P(-x,k)+\frac{\theta(-x)}{N}P(x,k)\\&+\frac{N-c-1}{N}P(x,k)+P(x,k)\left(\frac{c+x}{2N}(1-f_+)\right.\\&\left.+\frac{c-x}{2N}(1-f_-)\right)+P(x+2,k)\frac{c+x+2}{2N}f_+\\&+P(x-2,k)\frac{c-x+2}{2N}f_-.\end{aligned} \tag{21}$$

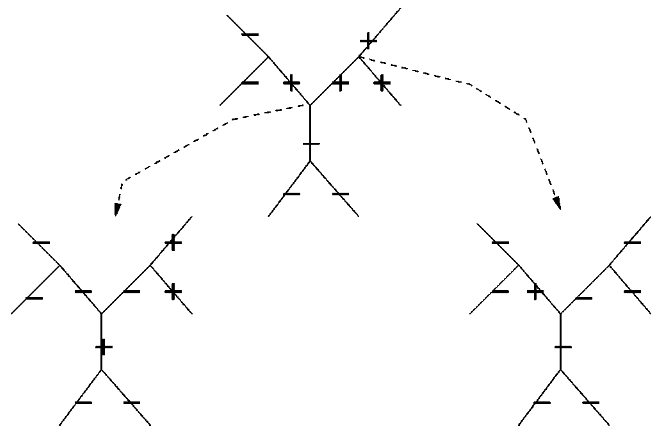

FIG. 2. An example of the falling dynamics for a spin system with $c=3$. The symbols $+/-$ indicate unsatisfied and satisfied bonds. The broken arrowed lines start at a spin that is flipped, and lead to the following configuration.

In this equation we have to define $\theta(0)$, the choice compatible with the conservation of probability is $\theta(0)=1/2$. Taking the limit $N\to\infty$ we may introduce the continuous time $\tau=k/N$, and obtain

$$\begin{aligned}\frac{dP(x)}{d\tau}=&\theta(-x)P(-x)+\theta(-x)P(x)-(c+1)P(x)+P(x)\\&\times\left(\frac{c+x}{2}(1-f_+)+\frac{c-x}{2}(1-f_-)\right)\\&+P(x+2)\frac{c+x+2}{2}f_++P(x-2)\frac{c-x+2}{2}f_-.\end{aligned} \tag{22}$$

The average energy per site at time τ is then given by $E(\tau)=\frac{1}{2}\Sigma_x xP(x,\tau)$, and one can show that the above mean field equation (22) respects the exact identity for the evolution of the average energy per spin:

$$\frac{dE}{d\tau}=-2\sum_{x>0} xP(x,\tau). \tag{23}$$

The case where $c=2$ (the one dimensional case) is accessible to an analytical solution, and we proceed by defining

$$u(\tau)\equiv P(-2,\tau),$$

$$v(\tau)\equiv P(0,\tau),$$

$$w(\tau)\equiv P(2,\tau).$$

One finds that

$$f_-=0,\qquad f_+=\frac{2w}{v+2w}, \tag{24}$$

and the full mean field evolution equations become

$$\frac{du}{d\tau}=f_+v+w,$$

$$\frac{dv}{d\tau}=-f_+v+2f_+w, \quad (25)$$

$$\frac{dw}{d\tau}=-(2f_++1)w.$$

If we look for a stationary solution of Eqs. (24) and (25), we find $w=0$, which expresses the fact that when τ is infinite, the system is in a metastable state. To solve Eqs. (24) and (25), we introduce λ so that $v=\lambda w$. Then λ obeys the equation

$$\frac{d\lambda}{d\tau}=\lambda+2, \quad (26)$$

with the initial condition $\lambda(0)=v(0)/w(0)$. Then Eq. (25) becomes

$$\frac{du}{d\tau}=\frac{2\lambda w}{\lambda+2}+w,$$

$$\frac{dw}{d\tau}=-\frac{\lambda+6}{\lambda+2}w, \quad (27)$$

$$\lambda+2=(\lambda(0)+2)e^{\tau}, \quad (28)$$

and $w(\tau)$ and $v(\tau)$ are given by

$$w(\tau)=w(0)\exp\left(-\tau+\frac{4}{\lambda(0)+2}(e^{-\tau}-1)\right), \quad (29)$$

$$v(\tau)=-2w(\tau)+w(0)(\lambda(0)+2)\exp\left(\frac{4}{\lambda(0)+2}(e^{-\tau}-1)\right). \quad (30)$$

The probability to have a positive value for the local fields then goes to zero at infinite τ, as expected, and the limit of v is $v(\infty)=(v(0)+2w(0))e^{-4/[\lambda(0)+2]}$. If we consider the geometric initial conditions used in the previous exact calculation of E_f, the induced initial conditions are $u(0)=(1-a)^2$, $v(0)=2a(1-a)$, and $w(0)=a^2$. In this case we obtain $E_f=-1+v(\infty)=-1+2ae^{-2a}$ reproducing the exact result [Eq. (17)]. Tapping the system with a tapping probability p, starting from the values $\{u(\infty),v(\infty),w(\infty)\}$, we obtain the new *tapped* values $\{u'(0),v'(0),w'(0)\}$. Defining $q\equiv(1-p)$, the relations between the old and *tapped* probabilities are

$$u'(0)=(1-3pq)u(\infty)+pq\,v(\infty),$$

$$v'(0)=2pq\,u(\infty)+(1-2pq)v(\infty), \quad (31)$$

$$w'(0)=pq.$$

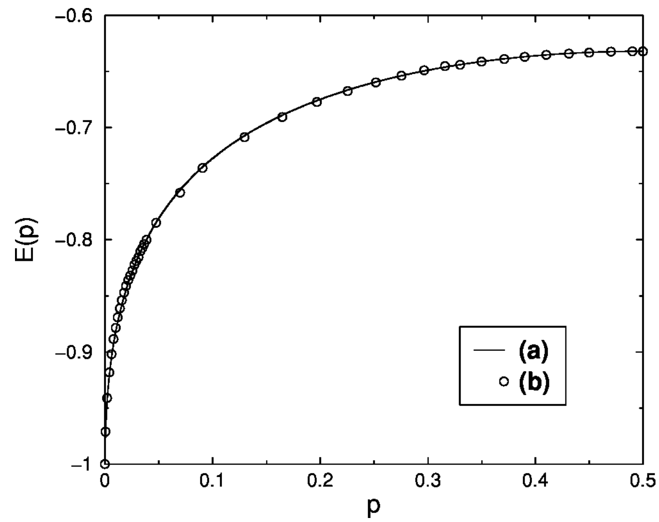

FIG. 3. Comparison between numerical simulations of tapping experiments (b) and the analytical result (a) obtained with Eq. (33).

Then, after another zero temperature evolution of the system, it reaches a new local energy probability distribution with $w'(\infty)=0$ and

$$v'(\infty)=(4pq+v(\infty)(1-4pq))\times\exp\left(-\frac{4pq}{4pq+v(\infty)(1-4pq)}\right). \quad (32)$$

At this stage of computation one should remark that this recursive equation contains one of the main features of our numerical simulations, that is, *reversibility*. Indeed, the process involved in Eq. (32) will reach an asymptotic value which is independent of the initial conditions and depends on p. Hence in the steady state regime under tapping, the probability $v_s(p)$ (the subscript s indicating steady state) for sites to have zero local field is solution of the fixed-point equation

$$v_s(p)=(4pq+v_s(p)(1-4pq))\times\exp\left(-\frac{4pq}{4pq+v_s(p)(1-4pq)}\right). \quad (33)$$

This equation can be solved numerically and the result is shown in Fig. 3 in comparison with the numerical simulations, which we see is excellent. The small p behavior of $E(p)$ from Eq. (33) is $E(p)=-1+\sqrt{2p}+O(p)$, indicating that in this case $\theta=1/2$.

Given the mean field nature of the above calculation we have used, we do not expect this approximation to correctly describe the approach towards the steady state. By direct comparison with the numerical simulations we have verified that this is indeed the case. Let us remark here that a defect of the mean field approximation scheme is that it cannot distinguish between a spin glass and a ferromagnet; this is clearly not a problem for the one dimensional situation where the two are identical.

TABLE I. Comparison of the numerical values of E^* and $E(p_c^+)$ for different values of the local connectivity c. The result for E^* when $c=3$ is a truncation of the analytical value $-15/14$.

c	p_c	$E(p_c^+)$	E^*	E_f	E_f^{MF}
3	0.25 ± 0.005	-1.076 ± 0.005	-1.0714	-1.045	-1.023
4	0.255 ± 0.005	-1.07 ± 0.005	-1.07 ± 0.01	-1.01	-1.005
5	0.45 ± 0.01	-1.4 ± 0.005	-1.4 ± 0.01	-1.396	-1.368

IV. HIGHER CONNECTIVITIES

The systems which we study are $\pm J$ spin glasses or uniform ferromagnets on random graphs with a fixed connectivity c. Let us first recall some analytical results of Refs. [2,3]. It was found that the mean number of metastable states increases exponentially with the number of sites in both cases. In addition, in Ref. [3] an annealed approximation to the Edwards entropy per spin of metastable states at fixed energy E was carried out,

$$s_{Edw}(E)=\ln(\langle N_{MS}(E)\rangle)/N, \tag{34}$$

which may be exact for the ferromagnet as the calculated entropy is always positive. Moreover, there is an energy threshold E^* above which the results are the same for the $\pm J$ spin glass and the ferromagnet; below, the ferromagnet has more metastable states and a nonzero magnetization. Hence, as far as the energy density of metastable states is concerned, both the ferromagnet and spin glass are the same above E^*—that is, the effect of loop frustration is negligible. In this regime, one also suspects that the zero temperature dynamics is the same. In particular, numerical simulations with 100 samples of $N=10\,000$ sites for connectivities of 3, 4, and 5 have found the same E_f for the spin glass and ferromagnet with very good accuracy (the relative error is about 10^{-6}). The results are show in Table I.

The result of tapping experiments on the systems with $c=3$ is displayed in Fig. 4. There is some critical tapping rate p_c above which the curves of $E(p)$ versus p are the same for the spin glass and the ferromagnet. Moreover, the ferromagnet is subject to a phase transition under tapping dynamics at p_c such that for $p<p_c$, the steady state reached is the ground state. Finite size effects have been studied, and revealed that the transition is first order [in as far that $E^{FM}(p_c^+)\neq E^{FM}(p_c^-)$], in contrast to the usual thermodynamic ferromagnetic transition in these systems which is second order [10]. For the ferromagnet in the region close to p_c one finds an excellent scaling of the energy as a function of N, $E^{FM}(N,p)=f(N(p-p_c))$, as shown in the inset of Fig. 4. This scaling may be used to optimize the determination of p_c. The first order nature of the transition may be seen explicitly by looking at the histogram over time (in the steady state regime) of the average energy per spin at $p=p_c$; in Fig. 5 one sees two separated peaks in the distribution, and not a single peak which splits into two as one would expect for a second order transition. Near p_c, for systems of finite size, there is therefore a coexistence of the two phases. The time dependence of the average energy per spin in the simulation leading to the histogram Fig. 5 is shown in Fig. 6. One sees that the system tunnels between the two coexisting states. The typical time for this tunneling increases as the system size increases, indicating, in thermodynamic language, a *free energy* barrier between the two phases. As the system size is increased, the occupation of the intermediate states of energy between the two phase of energy E_{GS} and $E(p_c^+)$ (between the two peaks in Fig. 5) is suppressed.

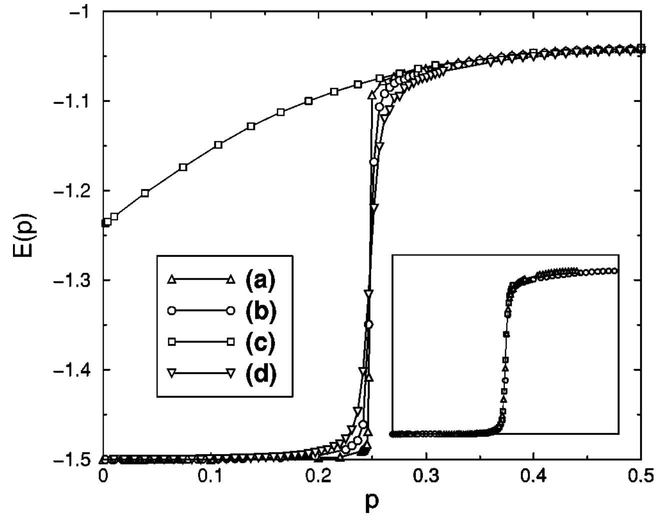

FIG. 4. Numerical simulations of tapping experiments for the spin glass (c) and the ferromagnet [(a), (b), and (d)] for $c=3$ for $N=1000$ [(c) and (d)], $N=2000$ (b), and $N=10\,000$ (a). The inset shows the scaling $E^{FM}(N,p)=f[N(p-p_c)]$ for $p\simeq p_c$ for $N=400$, 1000, and 2000.

We have also measured $E(p)$ by studying single systems of very large size ($N=10^6$); the results are shown in Fig. 7. Here again above p_c the curve for the spin glass and the ferromagnet are completely indistinguishable, and the ferromagnet reaches the ground state below p_c. For such large sizes, one no longer sees a coexistence of two phases around p_c as presumably the tunneling time has become much larger than the simulation time.

Moreover, for $N=10^6$, for $p>p_c$ the full temporal plots (and not just the steady state values) of $E^{SG}(p,t)$ and

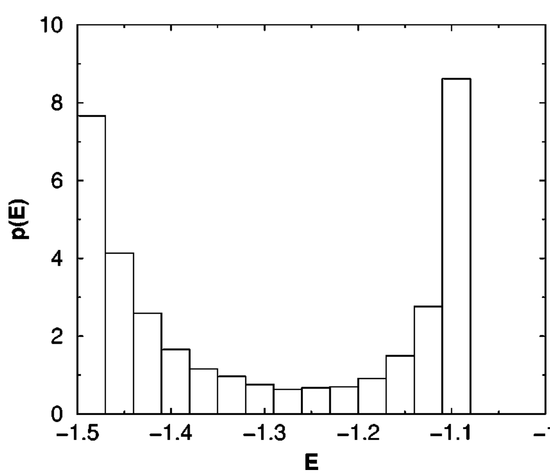

FIG. 5. Histogram of the energy per spin obtained from Fig. 6.

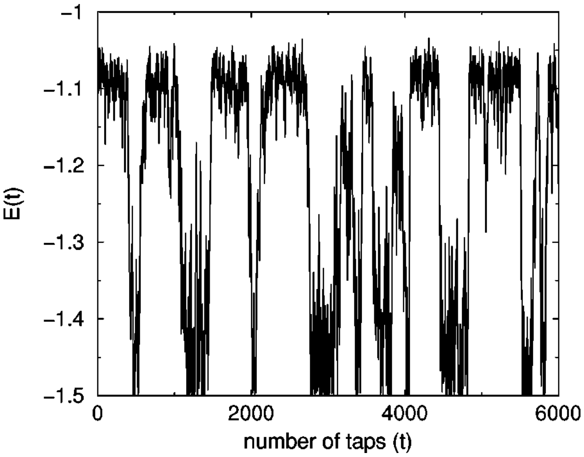

FIG. 6. Single run for a ferromagnet of local connectivity $c=3$, for $N=10\,000$ at $p=0.249\approx p_c$. One sees that because the size is not too large, the energy switches between two values, one not far from that of the ground state E_{GS} and the other not far from E^*.

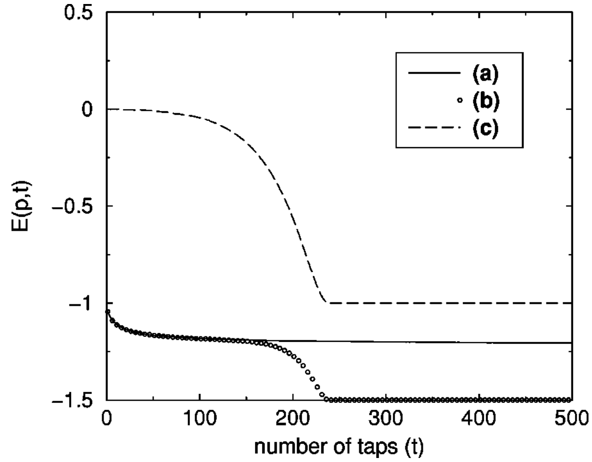

FIG. 8. Comparison of energy vs time (number of taps) for the $\pm J$ spin glass (a) and the ferromagnet (b) for $N=10^6$ spins at $p=0.05$. We display the magnetization for a ferromagnet (c) whose absolute value increases with time, whereas the magnetization of the spin glass remains zero.

$E^{FM}(p,t)$ (where t is the number of taps) are indistinguishable. For $p<p_c$ the two curves are identical up to a time t_{dif}, which depends on the initial configuration and the sequence of spins flipped during the tapping process; thay diverge after t_{dif}, when the ferromagnetic system reaches quickly the ground state (see Fig. 8). Once the ferromagnetic system has broken the Z_2 symmetry, the easiest way to lower the energy is to flip the spins which are opposed to the global magnetization (because they are more probable not to be in the direction of their local field), until all the spins are -1 or $+1$.

An identical behavior was found in systems with $c=4$ and 5. The comparison of $E^{SG}(p)$ and $E^{FM}(p)$ for $c=4$ is shown in Fig. 7. We remark that if we compare the different values of p_c when increasing c, and considering only odd (or even) connectivities, we find that p_c grows; we expect that it goes to 1/2 when c is very large, as the metastable states are more and more magnetized when c grows (in the case of the fully connected ferromagnetic Ising model, only the two ground states are metastable, so $p_c=1/2$).

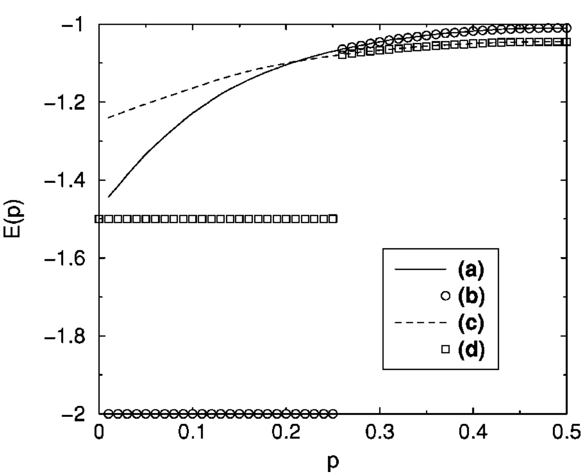

FIG. 7. Numerical simulations of tapping experiments for $c=3$ [(c) spin glass, (d) ferromagnet] and $c=4$ [(a) spin glass, (b) ferromagnet]. Here we compute the asymptotic energy for only one sample of very large size $N=10^6$. The results are quite similar to those for $N=10^5$, and do not change if we average over several samples; this indicates that, at these sizes, we are very near the thermodynamic limit, and we can study only single runs to compute the energy.

The behavior of the spin glass systems is similar to that for the system with $c=2$. The steady state energy $E^{SG}(p)$ is a monotonically decreasing and continuous function of p. For small p one finds that here $E^{SG}(p)\sim E^{SG}(0)+Ap$, giving $\theta=1$, in contrast to $\theta=1/2$ in the one dimensional case.

A tentative explanation for the ferromagnetic transition is as follows. In Ref. [3] it was also shown that for a ferromagnet the Edwards entropy as a function of E is concave for $E>E^*$ and convex for $E<E^*$. The value of $E(p_c^+)$ obtained from the tapping experiments are very close to those obtained for E^* in Ref. [3], the energy at which s_{Edw} becomes convex. The results are shown in Table I. Encouraged by this striking observation, we will try to make a tentative link with a possible thermodynamics for such systems. If we imagine that the energy of the system is governed by a partition function inspired by the flat Edwards measure over metastable states [1,15],

$$Z=\int dEN_{MS}(E)\exp(-N\beta E), \tag{35}$$

where β is a Lagrange multiplier corresponding to the inverse Edwards temperature which depends solely on p and not on E, and is a monotonically decreasing function of p for $p\in[0^+,1/2]$. The monotonicity hypothesis is supported by the simulation results that $E(p)$ decreases with decreasing p. Clearly, the energy which dominates in the sum is that obeying

$$\frac{\partial s_{Edw}(E)}{\partial E}-\beta=0. \tag{36}$$

However, if this saddle gives a true maximum of the action one must have also that

$$\frac{\partial^2 s_{Edw}(E)}{\partial E^2}<0, \tag{37}$$

and hence the Edwards entropy must be concave for the energy considered to be thermodynamically stable. Hence, for $E<E^*$, this suggests that the only stable energy is the ground state, and that the intermediate energies betweem E^* and E_{GS} are not realizable as steady state energies.

Let us mention here that we measured the energy in the simulations over a few hundred time steps after the energy appeared to stop to decay. To be sure that the systems considered here were in a stationary regime (and that the energy was not decaying extremely slowly as a function of time) we measured the correlation function at different waiting times t_w (the number of taps after the initial preparation of the system); that is to say,

$$C(t+t_w,t_w)=\frac{1}{N}\sum_{i=1}^{N} S_i(t_w)S_i(t_w+t). \tag{38}$$

In the stationary regime this should be a function only of t. In out-of-equilibrium systems the fact that the system is not in equilibrium shows up strongly as aging in the correlation function, i.e., $C(t+t_w,t_w)\neq C(t)$ (see, Ref. [24], and references within), even though the energy may be decaying so slowly that it appears to have reached its asymptotic equilibrium value. The time translational invariance of $C(t+t_w,t_w)$ is thus quite a rigorous test of whether the steady state regime has been attained. For example for the case $p=0.02$, with the waiting times $t_w=30\,000$ and 60 000 shown in Fig. 9, one sees clearly that after the appropriate translation of the t axis, the two functions collapse perfectly onto one another. One also sees that the decay of $C(t)$ (in the longtime regime we can now eliminate the t_w dependence) is exponential at large t and also that $C(t)$ decays to zero, indicating a form of ergodicity in the system. As pointed out in Ref. [14], this behavior of the correlation function seems a necessary condition for the validity of the scenario of Edwards, that under tapping all metastable states satisfying the relevant macroscopic constraints (fixed energy and compactivity) are equiprobable in the stationary regime of gently tapped or perturbed system.

In order to test the accuracy of the mean field approximation at high energies (where the system does not distinguish between the ferromagnet and the spin glass), we have compared the value of E_f obtained in the numerical simulations with the result E_f^{MF} obtained by numerical integration of Eq. (22). The comparison is shown in Table I, and we see that the agreement is quite good.

Finally we mention that we have also examined the reversibility of the tapping mechanism. If the system is tapped for a sufficiently long time, compatible with the relaxation

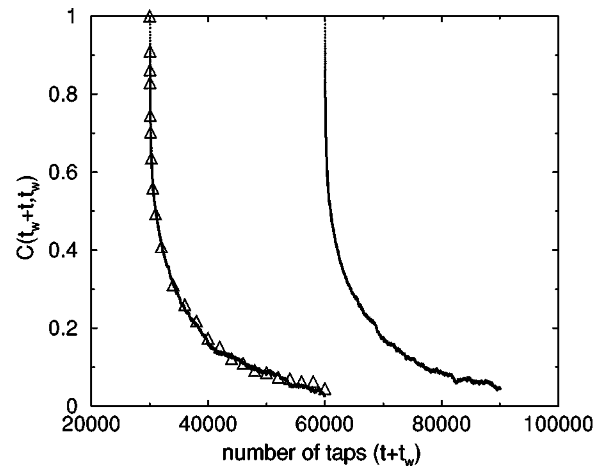

FIG. 9. Correlation function for the $\pm J$ spin glass with local connectivity $c=3$, vs the number of taps t for two values of the waiting time t_w: $t_w=30\,000$ and $t_w=60\,000$. The tapping value is $p=0.02$. The system contains $N=1000$ spins, and we have averaged over 1000 samples. Triangles show the right part of the curve, which corresponds to $t_w=60\,000$; shifted to the left by 30 000, it superimposes perfectly over the curve for $t_w=30\,000$, demonstrating the time translation invariance of the correlation function.

times discussed above, the system is completely reversible. This reversibility was found in the experiments in Ref. [5] once the system had left the initial fluffy state.

V. CONCLUSION

Granular media are a natural example of systems having an extensive entropy of metastable states. In such systems the role of thermal fluctuations are negligible, and in order to evolve one must apply some external tapping mechanism. One would ultimately like to be able to formulate some sort of thermodynamics for such systems. The proposition of Edwards [1] for a thermodynamics of such systems is an important step in this direction, and has had some success [14,17], but it has been shown not to be generically true [14]. A more general understanding of the asymptotic states of tapped systems has far reaching implications for computer science, as the tapping mechanism studied here is similar to certain algorithms used in optimization problems.

We have presented what appears to be an exact calculation of the steady state energy of a tapped one-dimensional spin glass or ferromagnet. For this problem we have obtained fixed point equations for the distribution of local fields under tapping. These equations also explain the reversibility observed in the numerical simulations. In a wide context of models we confirm the observations of Refs. [5,6,9], that, if one reduces the *strength* of tapping, then the compaction process, corresponding here to the reduction of the energy of the system, becomes more efficient. The existence of a first order type phase transition for tapped ferromagnets on random thin graphs is of great interest; a possible explanation using the calculations of Ref. [3] on the Edwards entropy for this system indicates the possibility that one may eventually construct a more general theory for the thermodynamics and even phase transitions in tapped systems. One is tempted to speculate that generically the convexity of the Edwards entropy below a certain energy threshold (denoted here by E^*) leads to a *collapse* to the ground state energy E_{GS}, the metastable states in the intervening energy values being unable to support a stable thermodynamics. In terms of granular media this sort of transition would correspond to a transition between a random close packed state to a crystalline close packed state. It would be interesting to find other systems (both theoretical and experimental) showing the same collapse phenomena in order to test this idea.

Finally let us mention some open questions, posed by this study, that we believe to be of interest for future investigation. Clearly a general goal would be, in the spirit of Edwards, to develop a thermodynamics to describe the stationary regime of tapped systems such as those studied here. The exact results presented here for one-dimensional systems provide a completely analytical understanding of the tapping dynamics which one may be able to rederive from static considerations. Indeed it was shown [17] in this simple context that several steady state observables may be predicted using the Edwards' measure. The phase transition found in the case of the ferromagnetic systems studied here is extremely interesting. An analytical understanding of this phenomena would be desirable; perhaps there exists a percolation type argument which would allow one to evaluate p_c. Also of interest is the decay of a system toward its final steady state energy. The slow logarithmic decay described by Eq. (2) has been used successfully to fit the experimental data of Ref. [5] and the simulation data of Refs. [9,12]. Our preliminary study of finite connectivity spin glasses and the SK model [18] indicates the presence of a slow dynamical regime for small values of the tapping parameter p, which is also compatible with a slow logarithmic decay, but the curves can also be well fitted by power law decays (with the same number of fitting parameters). However, there exist phenomenological arguments [11] and exact calculations and simulations on toy models [11–13] which support logarithmic decay.

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